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Hessian Matrix Used for Stellarator Coil Design and Error Fields Prediction

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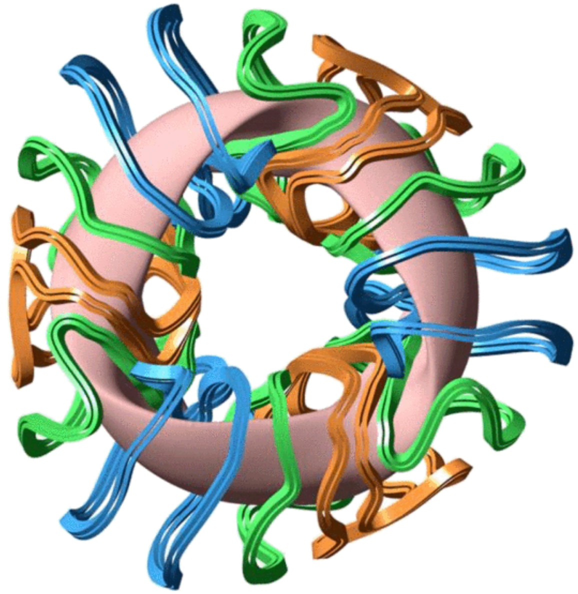
Introduction & Motivations

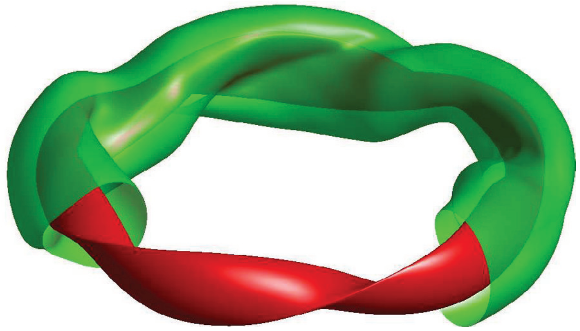
- **Stellarator coil design is a crucial challenge, even partly causing**
  - Termination of NCSX (*Neilson, et al., IEEE, 2010*)
  - Delay of W7-X construction (*Riße, FED, 2009*)
- **Two main approaches have been developed to design stellarator coils.**
  - **Surface current approximation**
    - NESCOIL (*Merkel, NF, 1987*)
    - NESVD (*Pomphrey, et al., NF, 2001*)
    - REGCOIL (*Landreman, NF, 2017; P1-14, Today*).
  - **Nonlinear optimization**
    - ONSET (*Drevlak, FST, 1998*)
    - COILOPT (*Strickler, et al., FST, 2002*)
    - COILOPT++ (*Breslau, et al., EPR2013*)
- **All the methods* require a pre-supposed "winding surface". Additional optimizations for the winding surface are needed.**
- **Nonlinear optimizations suffer the speed and robustness.**

  e.g. sensitive to initial guess; bad convergence performance; get trapped to local minimum.

** ONSET takes two reference surfaces and interpolates between them.*



Modular coils (non-planar) and the plasma for NCSX (*Neilson, et al., IEEE, 2010*)



A winding surface (green) and the reference plasma boundary (red) for W7-X (*Landreman, NF, 2017*)

Description of the new method*

* *Zhu, C., Hudson, S. R., Song, Y., & Wan, Y. (2017). New method to design stellarator coils without the winding surface. Nuclear Fusion (accepted), arXiv:1705.02333.*

Flexible Optimized Coils Using Space-curves (FOCUS) code

❖ Using three-dimensional representation to describe general coils.

Fourier series are used for representing coils in the cartesian coordinate (x, y, z)

$$x(t) = X_{c,0} + \sum_{n=1}^{N_F} \left[X_{c,n} \cos(nt) + X_{s,n} \sin(nt) \right]$$

$t \in [0,2\pi]$ *is an arbitrary parameter.*

A coil is fully determined by the its Fourier coefficients. All the free variables are

$$\mathbf{X} = \left[\overbrace{\underbrace{X^1_{c,0}, \cdots, X^1_{c,N}}_{\text{N+1}}, \underbrace{X^1_{s,1}, \cdots, X^1_{s,N}}_{\text{N}}, Y^1_{c,0}, \cdots, Z^1_{s,N}, I^1}^{\text{6N+4}}, \cdots, \cdots, X^{N_c}_{c,0}, \cdots, I^{N_c} \right]$$

❖ Multiple objective functions are implemented to be optimized.

Coil parameters are varied to satisfy multiple objective functions, including both physics requirements and engineering constraints. Objective functions can be **arbitrarily** constructed. FOCUS has implemented minimum required constraints and retained the **maximum flexibilities**.

$$\min_{\mathbf{X} \in \mathbb{R}^n} F(\mathbf{X}) = \sum_i w_i f_i(\mathbf{X})$$

- ✓ **normal magnetic field error**

$$f_B(\mathbf{X}) \equiv \int_S \frac{1}{2} (\mathbf{B} \cdot \mathbf{n})^2 \, ds$$

- ✓ **toroidal flux**

$$f_\Psi(\mathbf{X}) \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left(\frac{\Psi_\zeta - \Psi_o}{\Psi_o} \right)^2 d\zeta$$

- ✓ Fourier spectrum of magnetic field

$$f_H(\mathbf{X}) \equiv \frac{1}{2} \sum_{m,n} w^H_{mn} \left| \Delta_{mn} - \Delta^o_{mn} \right|^2$$

- ✓ **coil length penalty**

$$f_L = \frac{1}{N_C} \sum_{i=1}^{N_C} \frac{e^{L_i}}{e^{L_{i,o}}} \qquad f_L = \frac{1}{N_C} \sum_{i=1}^{N_C} \frac{1}{2} \frac{(L_i - Li,o)^2}{L^2_{i,o}}$$

- ✓ coil-coil separation

$$f_C = \sum_{i,j} \int_{C_i} \int_{C_j} \frac{dl_i \, dl_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}$$

- ✓ spectral condensation

$$f_A = \frac{1}{2} \sum_i^{N_C} \int_0^{2\pi} (x'' x' + y'' y' + z'' z')^2 \, dt$$

FOCUS calculates the first & second derivatives analytically.

Many nonlinear optimization algorithms require the gradient or even the Hessian. Usually, it's hard to differentiate the derivatives, especially for complex problems.

❖ The derivatives in FOCUS are calculated (semi-)analytically.

The Fourier representation and all the objective functions are differentiable. Rather than using finite difference, we analytically derive the gradient and Hessian, to improve the speed and accuracy.

For instance, to calculate the first and second derivatives of the normal field error,

$$f_B(\mathbf{X}) \equiv \int_S \frac{1}{2} (\mathbf{B} \cdot \mathbf{n})^2 \, ds$$

For an arbitrary variable, $\forall X_m \in \mathbf{X}$, we have

$$\frac{\partial f_B}{\partial X_m} = \int_S (\mathbf{B} \cdot \mathbf{n}) \left(\frac{\partial \mathbf{B_V}}{\partial X_m} \cdot \mathbf{n} \right) ds \qquad \frac{\partial^2 f_B}{\partial {X_m}^2} = \int_S \left(\frac{\partial \mathbf{B}_V}{\partial X_m} \cdot \mathbf{n} \right)^2 + \left(\frac{\partial^2 \mathbf{B}_V}{\partial {X_m}^2} \cdot \mathbf{n} \right)$$

The magnetic field produced by external coils is a functional of coil geometries $\boldsymbol{x}(\boldsymbol{X})$

$$\mathbf{B}_V(\bar{\mathbf{x}}) = \frac{\mu_0}{4\pi} \sum_{i=1}^{N_C} I_i \int_{C_i} \frac{d\mathbf{l}_i \times \mathbf{r}}{r^3} \qquad \frac{\partial \mathbf{B}_V}{\partial X_m} = \int_0^{2\pi} \frac{\delta \mathbf{B}_V}{\delta \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial X_m} dt$$

$$\frac{\partial^2 \mathbf{B}_V}{\partial {X_m}^2} = \int_0^{2\pi} \frac{\delta^2 \mathbf{B}_V}{\delta x^2} \left(\frac{\partial x}{\partial X_m} \right)^2 + \frac{\delta^2 \mathbf{B}_V}{\delta x \delta y} \frac{\partial y}{\partial X_m} \frac{\partial x}{\partial X_m} + \frac{\delta^2 \mathbf{B}_V}{\delta x \delta z} \frac{\partial z}{\partial X_m} \frac{\partial x}{\partial X_m} dt$$

Different optimization algorithms are applied.

Applying the functional derivatives

$$\delta \mathbf{B}_V = \int_0^{2\pi} \left[\frac{3\mathbf{r}\cdot\mathbf{x}'}{r^5}\mathbf{r}\times\delta\mathbf{x} + \frac{2}{r^3}\delta\mathbf{x}\times\mathbf{x}' + \frac{3\mathbf{r}\cdot\delta\mathbf{x}}{r^5}\mathbf{x}'\times\mathbf{r} \right] dt.$$

$$\begin{aligned}\delta^2 \mathbf{B}_V = \int_0^{2\pi} & \frac{3\left[\mathbf{r}\cdot\delta^2\mathbf{x} + 5(\mathbf{r}\cdot\delta\mathbf{x})^2 - \delta\mathbf{x}\cdot\delta\mathbf{x}\right]}{r^5}(\mathbf{x}'\times\mathbf{r}) \\ & + \frac{3\left[\mathbf{r}\cdot\delta\mathbf{x}' + 5(\mathbf{r}\cdot\mathbf{x}')(\mathbf{r}\cdot\delta\mathbf{x}) - \delta\mathbf{x}\cdot\mathbf{x}'\right]}{r^5}(\mathbf{r}\times\delta\mathbf{x}) \\ & + \frac{3\mathbf{r}\cdot\delta\mathbf{x}+6}{r^5}(\delta\mathbf{x}\times\mathbf{x}') + \frac{3\mathbf{r}\cdot\delta\mathbf{x}}{r^5}(\delta\mathbf{x}'\times\mathbf{r}) \\ & + \frac{3\mathbf{r}\cdot\mathbf{x}+6}{r^5}(\mathbf{r}\times\delta^2\mathbf{x}) + \frac{2}{r^3}(\delta^2\mathbf{x}\times\mathbf{x}') + \frac{2}{r^3}(\delta\mathbf{x}\times\delta\mathbf{x}')\end{aligned}$$

❖ **Several gradient-based or Hessian-based optimization algorithms are applied in FOCUS. More algorithms can be employed.**

Gradient-based algorithms

- Differential/Gradient Flow
- Nonlinear Conjugate Gradient

Hessian-based algorithms

- Modified Newton Method
- Hybrid Powell Method
- Truncated Newton Method

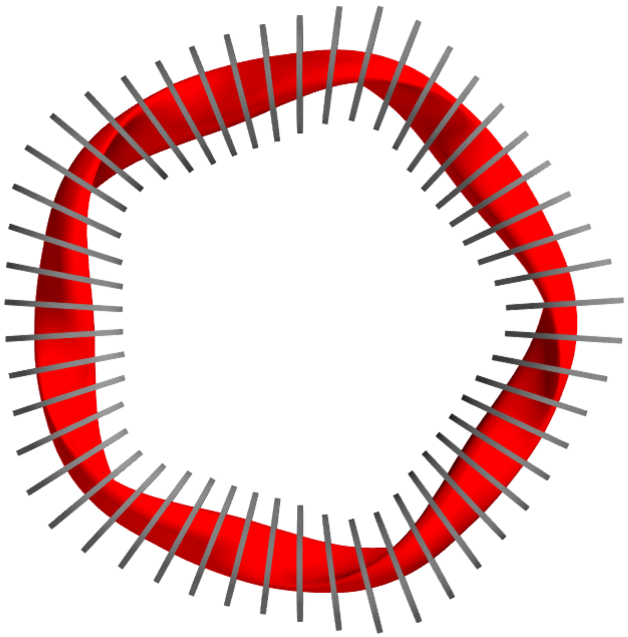
Numerical applications

Reproducing W7-X coils from initial circular coils.

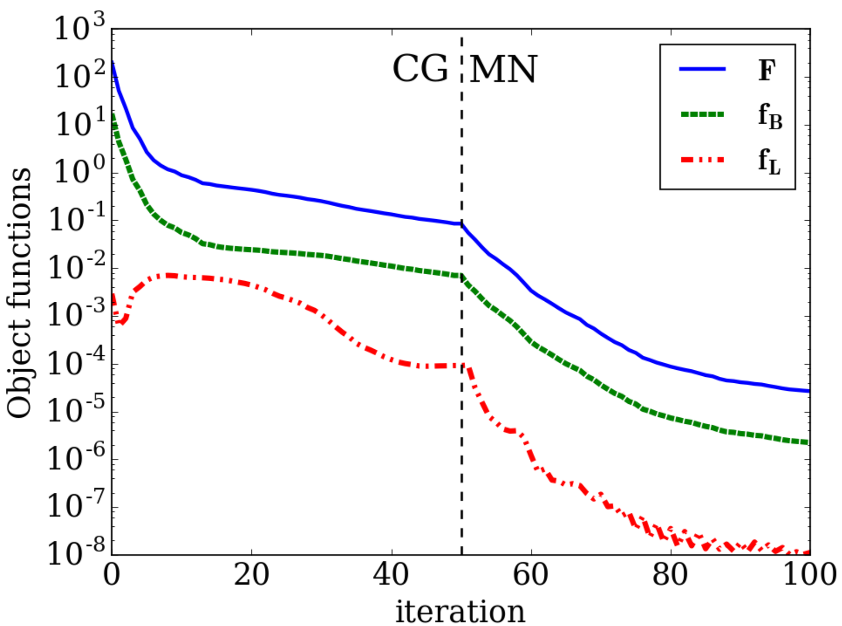
- **Target boundary:** LCFS from the reference standard configuration with known B_n distribution from the actual coils *(data courtesy of J. Geiger and the W7-X team)*;
- **Constraints:** minimize $F(\mathbf{X}) = w_b f_B + w_L f_L$

$$f_B(\mathbf{X}) \equiv \int_S \frac{1}{2}\left(\mathbf{B_{coils}} \cdot \mathbf{n} - T_{Bn}\right)^2 ds. \qquad f_L(\mathbf{X}) = \frac{1}{N_C} \sum_{i=1}^{N_C} \frac{1}{2} \frac{(L_i - Li, o)^2}{L_{i,o}^2} .$$

- **Initial guesses**: 50 circular coils (r = 1.25m) equally placed surrounding the plasma;
- **Optimizer**: 50 iterations Nonlinear Conjugate Gradient (CG) + 50 iterations Modified Newton method (MN), ~3.6h with 128 CPUs (without enforcing periodicity and stellarator symmetry)

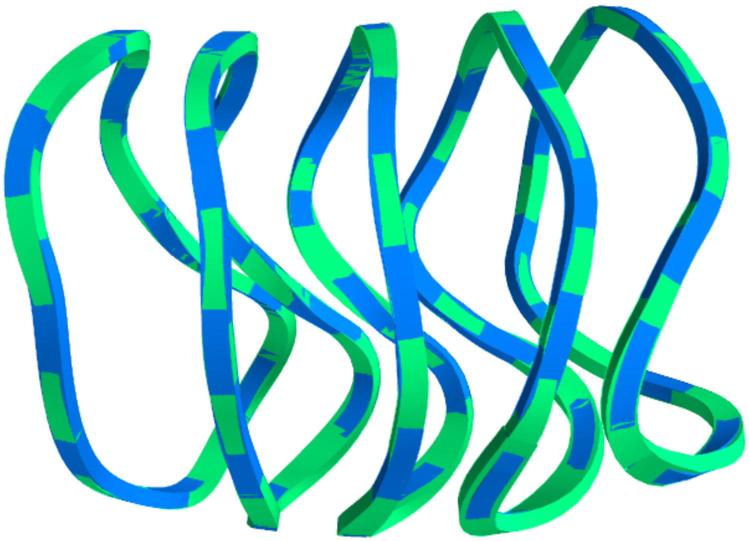


The initial circular coil (grey).

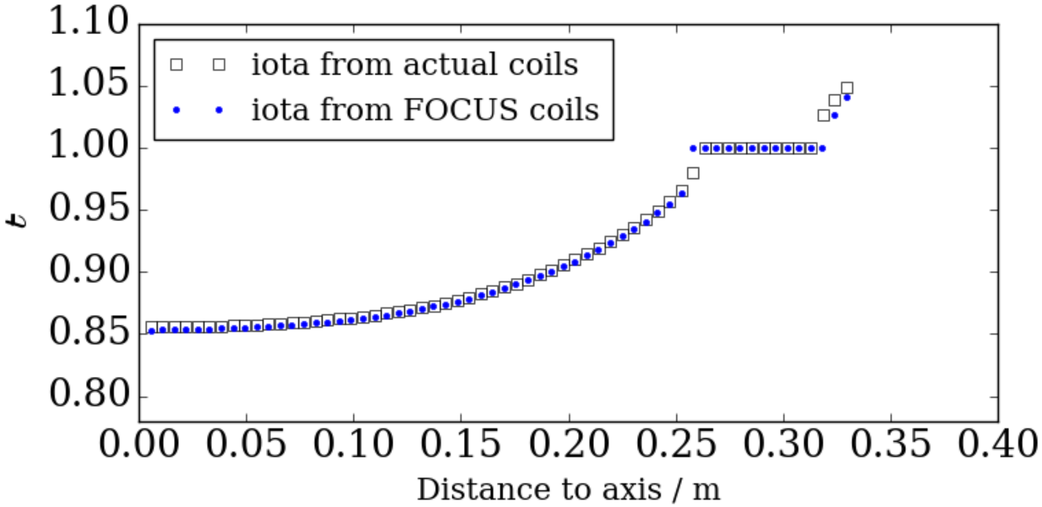


f_B decreases from $1.71{\times}10^{1}$ to $2.24{\times}10^{-6}$ and f_L is reduced from $2.83{\times}10^{-3}$ to $1.13{\times}10^{-8}$.

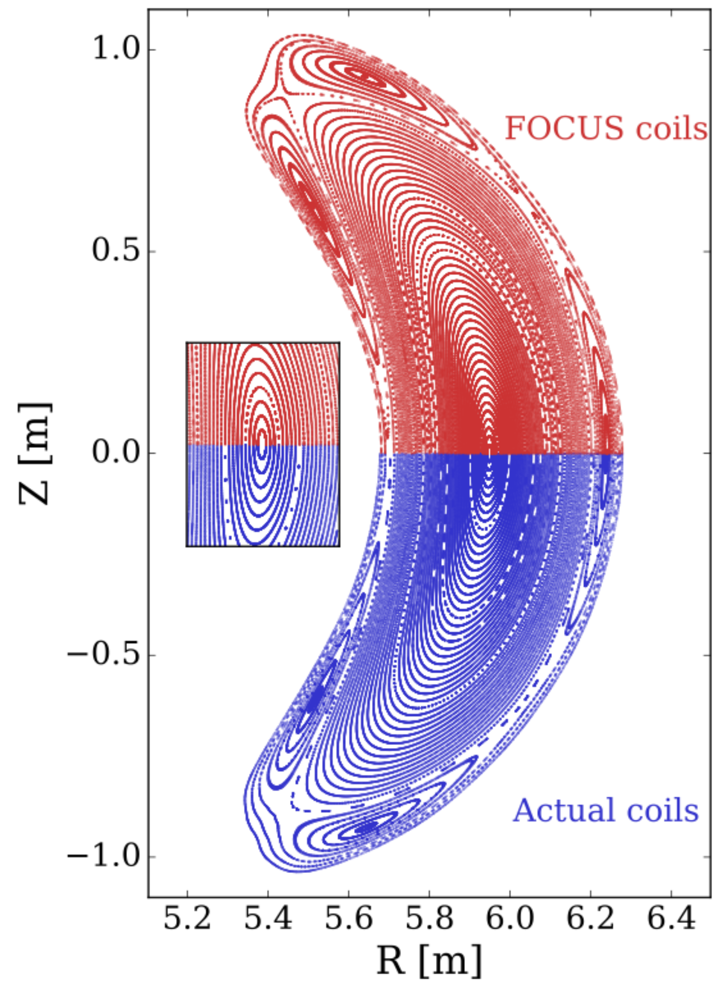
FOCUS gets almost the same coils and identical magnetic field.



Comparing optimized coils (green) and the actual coils (blue). Actual coils came from iterative NESCOIL runs with manually optimized winding surfaces.

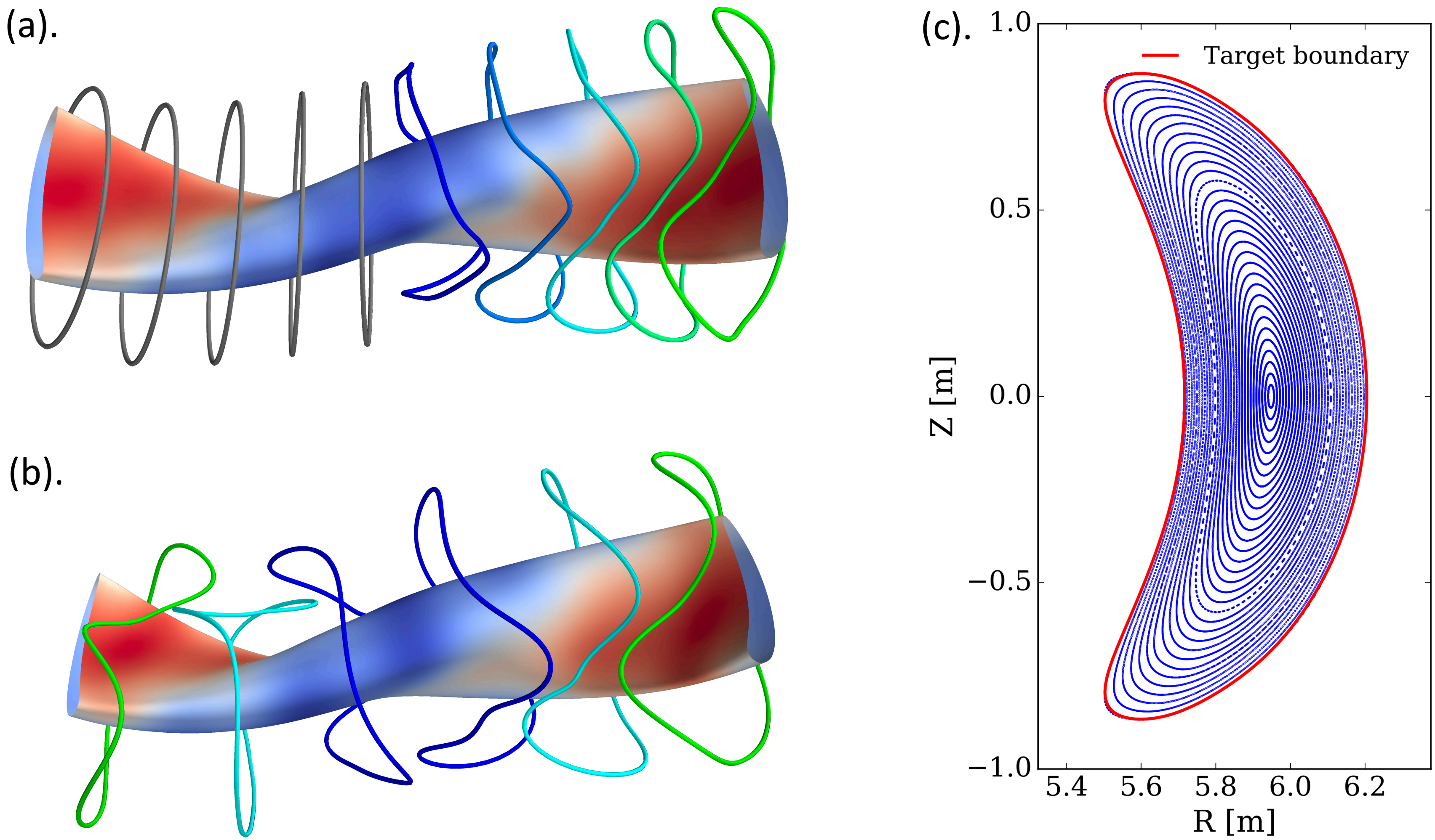


Comparison of the iota profiles (at the bean-shaped cross-section).



Poincare plots of the vacuum field produced by FOCUS coils (upper, red) and the actual coils (lower, blue).

Interesting study of reconstructing W7-X with as few as 30 coils.

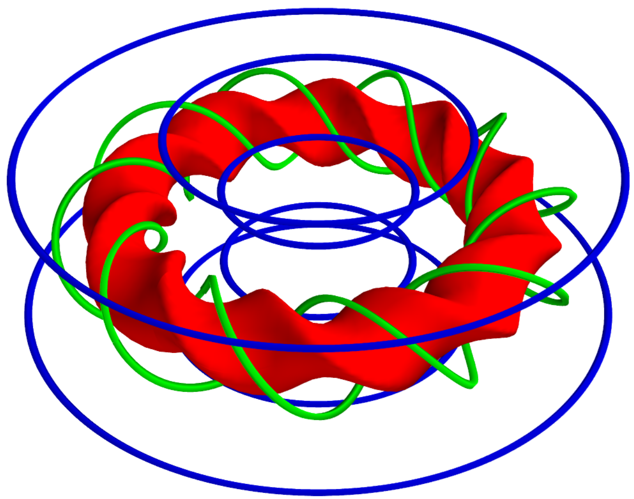


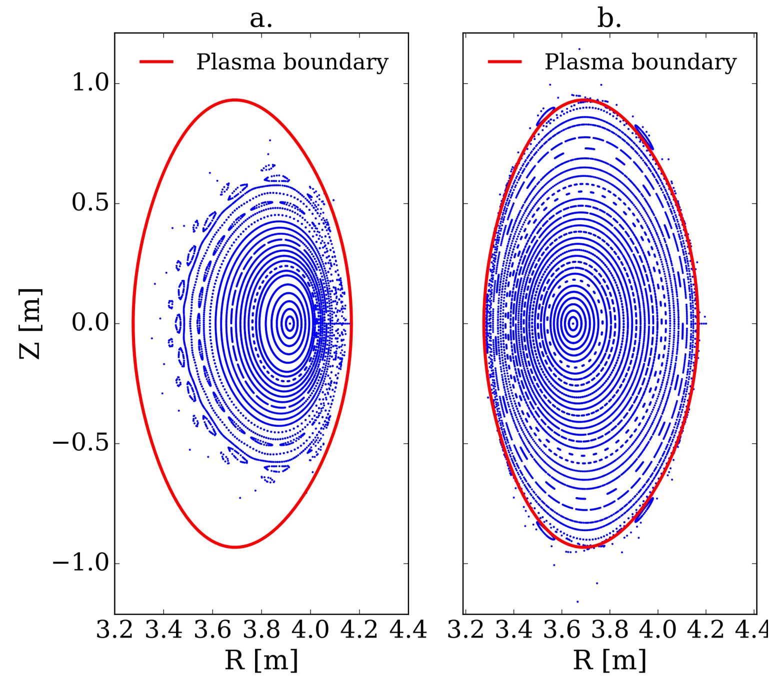
(a). FOCUS optimized 50 coils (half period initial coils and half final coils are plotted); (b). 30 coils solution; (c) Poincare plots of the vacuum field produced by 30 coils compared with the target plasma boundary. The color on the surfaces indicated the B magnitude produced by external coils. Further explorations on the magnetic ripple, transport, etc. should be carried out.

Optimize helical / modular coils for LHD.

❖ Using FOCUS to correct numerical parameterization errors.

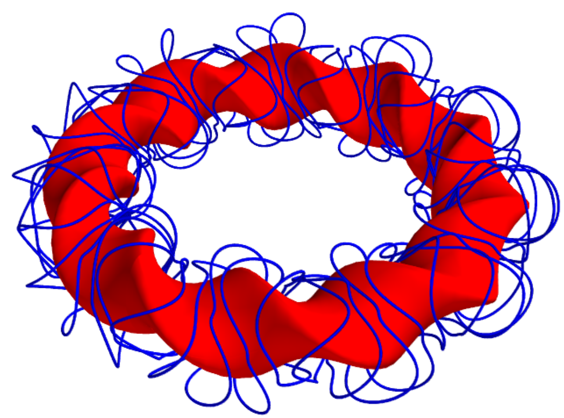
1. Fit the actual coils with Fourier series;
2. Truncated errors are introduced and result in bad approximated magnetic field;
3. Use FOCUS to optimize the “bad” coils and coil parameters are adjusted to reproduce the target magnetic field.

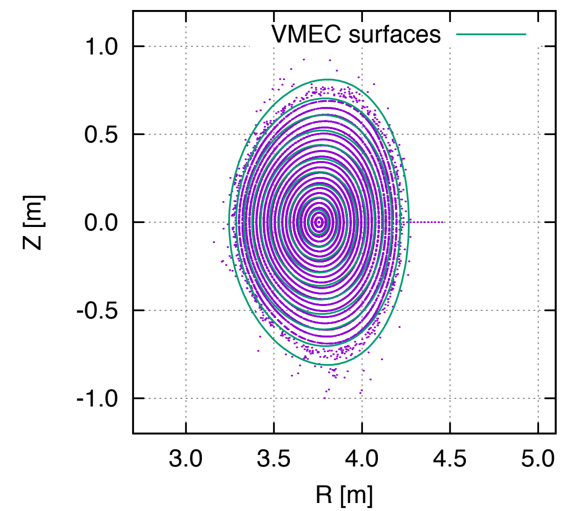




❖ Designing modular coils for LHD.

1. Initial results from Prof. Y. Suzuki (NIFS),
2. Starting with 50 circular coils.
3. Final magnetic field is acceptable, but coils are overlapped at the concave areas to form “helical” shapes;
4. More works are in progress. Results can be improved with better constraints.





Coil sensitivity analysis

Hessian method for error fields sensitivity analysis.

- **Error fields are unavoidable and should be carefully controlled. They are more critical for stellarator due to the complex coil shapes.**

- **To minimize error fields, the main challenge is the geometrical precision during coil manufacturing and coil assembly.**

W7-X has obtained remarkable achievements (*Pendersen, et al., NC, 2016; Lazerson, I-17, Thursday*).

- **With analytically calculated Hessian, FOCUS can rapidly analyze the sensitivities of coil displacement on error fields.**

If we have a function $F(\boldsymbol{X})$ for evaluating how well the produced magnetic field matches the target, the Taylor expansion tells that

$$F(\mathbf{X}+\delta\mathbf{X}) = F(\mathbf{X}) + g^T\delta\mathbf{X} + \frac{1}{2}\delta\mathbf{X}^T H \delta\mathbf{X} + \mathcal{O}(\delta\mathbf{X}^3)$$

Suppose that $\boldsymbol{X}$ is **near a local minimum $\boldsymbol{X_{min}}$** and **$\delta\boldsymbol{X}$ is small**, only the quadratic term is left. Any arbitrary coils displacements can be composed in eigen-space (eigenvalues $\boldsymbol{v}_i$ of H can form an orthogonal bias).

$$\delta\mathbf{X} = \sum_i^n a_i \mathbf{v}_i \quad |\delta\mathbf{X}| = \sqrt{\sum_i^n {a_i}^2} = \xi\,, 0 < \xi \ll 1$$

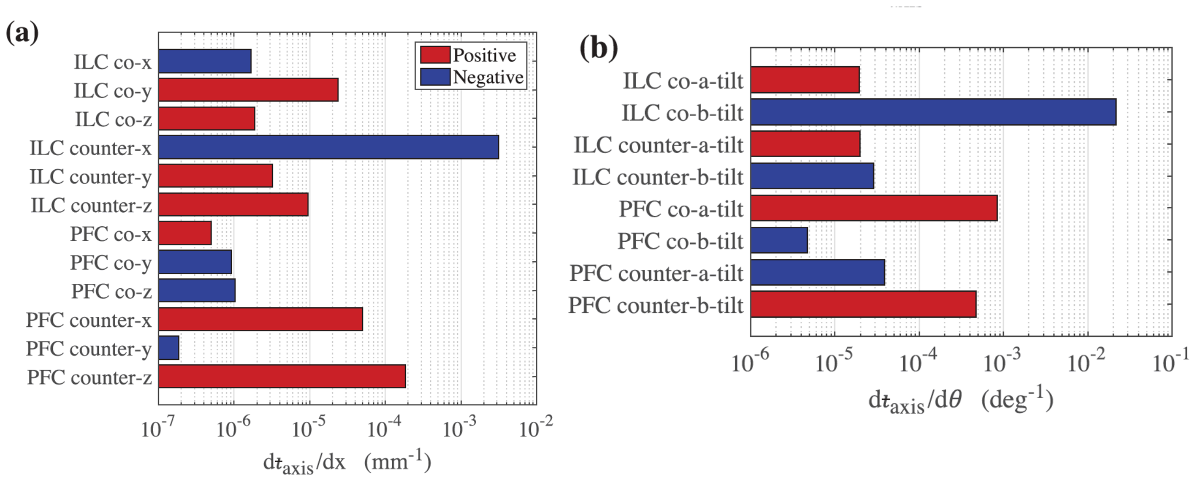
The maximum change in F (error field) happens with the displacement/deformation in the direction of eigenvector corresponding to the largest eigenvalue.

$$\delta F \approx \frac{1}{2}\delta\mathbf{X}^T H \delta\mathbf{X} = \frac{1}{2}\sum_i^n {a_i}^2 \lambda_i \le \frac{1}{2}\xi^2\lambda_m$$

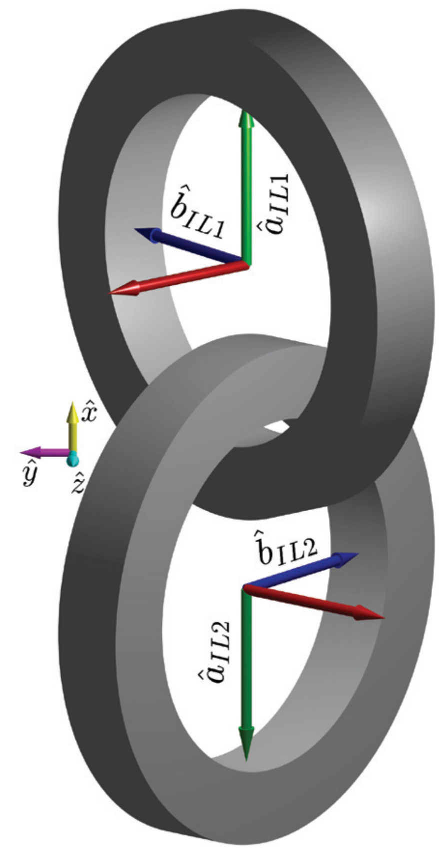
Previous study on error fields analysis for CNT coils.

CNT (*Pedersen et al., FST, 2004; Volpe, I-24, Friday*) is an interesting stellarator constructed at the Columbia University with two inter-linked (IL) circular coils and two Helmholtz coils.

- Hammond *et al.* (*PPCF, 2016; also at the 20th ISHW*) carried out an excellent study on analyzing the error fields of CNT. They calculated the derivatives of ι_{axis} with respect to manually-defined rigid displacements with finite difference.



- Conclusion: **The separation of the IL coils** (along the negative $\hat{x}$ axis) and **the tilt angle between the IL coils** (clock-wise around $\hat{b}$) have the greatest influences.



Ten defined rigid displacements for IL coils.

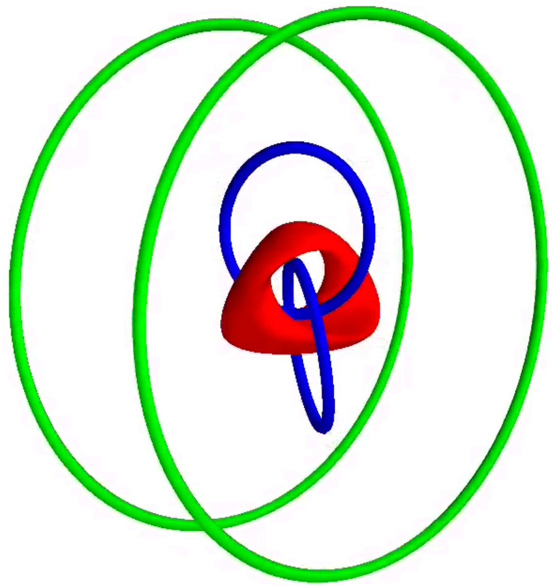
Procedures of using FOCUS to analyze coil sensitivities.

❖ Criteria for evaluating the error fields:

$$f_B(\mathbf{X}) \equiv \int_S \frac{1}{2} \left(\frac{\mathbf{B} \cdot \mathbf{n}}{|\mathbf{B}|} \right)^2 ds$$

❖ Find the minimum (coil optimization):

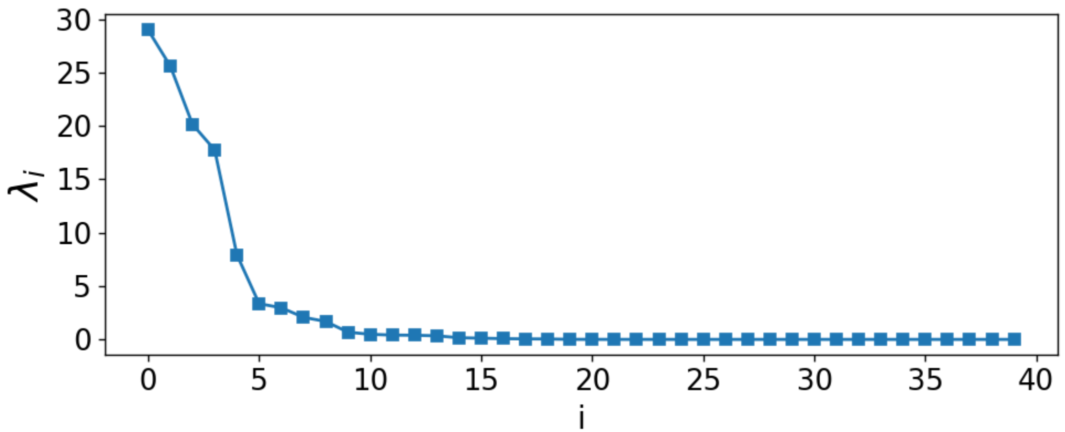
- starting from two circular IL coils with zero tilt angle;
- N_F=1 to enforce planar coils;
- use the Truncated Newton method to minimize f_B;
- $f_B = 10^{-5}$, $|g| = 10^{-14}$



CNT-like configuration and coils.

❖ Eigenvalue decomposition of the Hessian

- The Hessian is a 40×40 symmetric matrix.
- At the minimum, it's semi-positive definite.
- Eigenvalues decay rapidly.
- The maximum eigenvalue is about 28.99, with the second of 25.61.



Eigenvalues spectra of the Hessian matrix.

The most sensitive displacements are tilt and separation + ellipticity.

- **Apply displacements in the direction of the first two eigenvectors.**

  On the right are the perturbed coil geometries under different displacements. The upper is in the direction of eigenvector corresponding to the largest eigenvalue and the lower is the second largest.

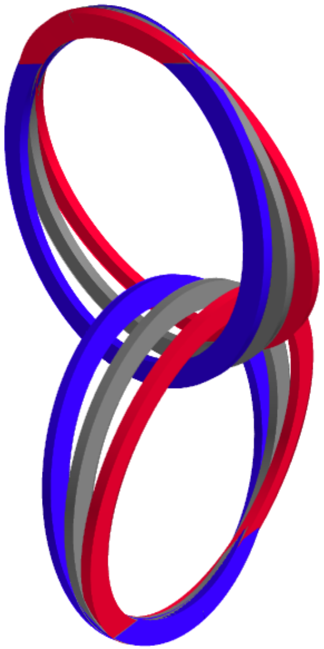
  - Grey: "equilibrium" coils;
  - Blue: $\xi = -0.5$;
  - Red : $\xi = \quad 0.5$.

- **Deep looks at the two displacements.**
  - **Upper figure:**

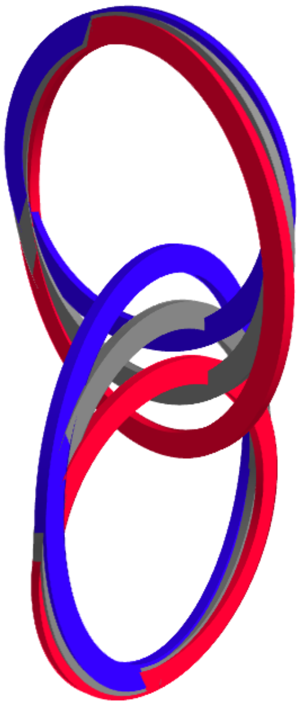
    The displacement is **changing the tilt angle**. The two IL coils are in different rotating directions (co-b + counter-co-b).
  - **Lower figure:**

    The deformation is **changing the separation and IL coils become elliptical**. Positive and negative deformations have the same effects.
  - **Only the interlinked parts matter.**

- **The insensitive displacements are mainly related to the Helmholtz coils.**



$X_{min} + \xi v_1$



$X_{min} + \xi v_2$

Summary & Future Work

❖ **Introduced a new method for designing stellarator coils, with two new features:**

- more flexible by getting rid of the "winding surface"
- more robust and faster by employing analytically calculated derivatives

❖ **Presented examples of using the new method to optimize stellarator coils.**

FOCUS can be used for various coil types. By only minimizing Bn and targeting the length, we can almost reproduce the W7-X actual coils. The possible solution of 30 coils is exciting.

❖ **Demonstrated an elegant method to analyze coil sensitivities using the Hessian method.**

CNT-like results coincide with previous work by using numerical calculations. Sensitivity analysis for better controlling error fields, reducing cost and saving time.

Undergoing work and future plans

- Applying FOCUS to design coils for different machines.
  - Improved coils for HSX (with A. Bader and T. Kruger)
  - LHD & LHD-like heliotrons (with Y. Suzuki)
  - RMP coils for DIII-D (with N. Logan)
- Adding global optimization algorithms and implementing more constraints.
- Coil sensitivity analysis on resonant harmonics, coupled with magnetic forces and applied for complex configurations (W7-X).